Incidence-Adjacent Vertex Distinguishing Equitable Total Coloring of Mycielski Graphs

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Abstract: The incidence-adjacent vertex distinguishing equitable total coloring of the Myceilski graphs of path, cycle, wheel and fan are discussed, and of which the incidence-adjacent vertex distinguishing equitable total chromatic numbers are confirmed efficiently by using constructive method and color adjusting technique base on the structure quality of the graphs. This method of the research provides an important reference for us to study the graph coloring problem with structural relations.

1. Introduction

The graph coloring problem proposed by network optimization and information technology is one of the important research contents of graph theory. In order to expand the application field of the coloring theory of graphs, the concepts of the incidence adjacent vertex distinguishing total coloring and the incidence adjacent vertex distinguishing equitable total coloring are put forward successively in[1,2]. The new coloring concept is widely concerned by the graph theorists. It was published that many results of the incidence adjacent vertex distinguishing total coloring total coloring of graphs such as spider graphs and fishing-net graphs in [3], crown graphs $C_m \cdot F_n$、$C_m \cdot S_n$ and $C_m \cdot W_n$ in [4], join graphs $P_m \vee F_n$, $P_m \vee W_n$ in [5], Mycielski graphs of path, fan and star in [6], some direct product graphs [7]. In [8] the incidence adjacent vertex distinguishing equitable total chromatic numbers of windmill graph K_3^t, $D_{m,4}$ and gear graph $\widetilde{W}_n$ are given.

In this paper, we research the incidence-adjacent vertex distinguishing equitable total coloring of Mycielski graphs of path, cycle, fan and wheel and of which the incidence-adjacent vertex distinguishing equitable total chromatic numbers are confirmed. And we will testify that the conjecture in [2] for Mycielski graphs is true, say, the incidence-adjacent vertex distinguishing equitable total chromatic numbers of the Mycielski graphs are not more than $\Delta(\mathrm{M}(G)) + 2$.

Suppose that the vertex s coloring set is $C(s) = \{\varphi(s)\} \cup \{\varphi(st) | st \in E(G)\}$ and the complementary set of $C(s)$ in $C = \{1, 2, \cdots, k\}$ is that $\overline{C}(s) = C \backslash C(s)$. For other unspecified definitions and notations such as the Myceilski graph of G in this article, we refer to Bondy and Murty[9].

Definition 1.1 [2] *Suppose φ is a k-I-AVDTC of simple connect graph $G(V,E)$ ($|V(G)| \geq 2$), satisfying*

for $\forall\, i,j \in \{1, 2, \cdots, k\}$, and $i \neq j$, we have

$\big||T_i| - |T_j|\big| \leq 1$,

then we call φ is the incidence-adjacent vertex distinguishing equitable total coloring of G (abbreviated to k-I-AVDETC), and call

$$\chi^i_{aet}(G) = min\{k|\ G\ o\varphi\ k\text{-}I\text{-}AVDETC\}$$

the adjacent vertex distinguishing equitable I-total chromatic number of G, *where* $T_i = V_i \cup E_i$, $V_i = \{s|s \in V(G),\ \varphi(s) = i\}$, *and* $E_i = \{e|e \in E(G), \varphi(e) = i\}$.

By the above definition, we obviously have

$$\chi^i_{aet}(G) \geq \chi^i_{at}(G) \geq \Delta(G),$$

where $\Delta(G)$ *is the maximum degree of* G.

Lemma 1.1 [2] *For a simple connected graph G with that* $|V(G)|\geq 2$, *if there are adjacent vertices with the same maximum degree, then*

$$\chi^i_{aet}(G) \geq \chi^i_{at}(G) \geq \Delta(G) + 1. \quad (1)$$

Conjecture [2] *Let G(V,E)be a simple graph, then*

$$\chi^i_{aet}(G) \leq \Delta + 2. \quad (2)$$

2. Main Theorems

Now we are ready to present our main theorems.

Theorem 2.1 *Suppose the Mycielski graph of a path* P_n *is* $\mathcal{M}(P_n)$, *then*

$$\chi^i_{aet}(\mathcal{M}(P_n)) = \begin{cases} n+1, & n = 2,3,4; \\ n, & n \geq 5. \end{cases} \quad (3)$$

Proof. Suppose the vertex set and edge set of $\mathcal{M}(P_n)$ are that $V(\mathcal{M}(P_n))= \{s_1,\ s_2, \cdots,\ s_n,\ t_1,\ t_2, \cdots,\ t_n, w\}$ and $E(\mathcal{M}(P_n)) = \{s_i s_{i+1},\ t_i w,\ t_n w | i = 1, 2, \cdots, n-1\} \cup \{s_i t_j | s_i s_j \in E(P_n), i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, n\}$, respectively. Allow us discuss the following four cases..

Case 1 If $n = 2$, owing to the fact that $\mathcal{M}(P_n) \cong C_5$, the conclusion is apparently true(see [2]).

Case 2 If $n = 3$, on account of the structure of $\mathcal{M}(P_n)$, we get that $\chi^i_{aet}(\mathcal{M}(P_n)) \geq 4$ according to lemma 1.1. To prove that $\chi^i_{aet}(\mathcal{M}(P_n)) = 4$, only we need to prove that $\mathcal{M}(P_n)$ has a 4-I-AVDETC.

Now construct the map from $T(\mathcal{M}(P_3))$ to $\{1, 2, 3, 4\}$ as follows

$\varphi(t_1) = 2, \varphi(t_2) = 1, \varphi(t_3) = 4, \varphi(w) = 1; \varphi(s_i) = i+1, i = 1, 2, 3;$

$\varphi(t_1 t_2) = 1, \varphi(t_2 t_3) = 4; \varphi(t_1 s_2) = \varphi(t_2 s_1) = 2, \varphi(t_2 s_3) = \varphi(t_3 s_2) = 3;$

$\varphi(s_1 w) = 3, \varphi(s_2 w) = 4, \varphi(s_3 w) = 1.$

The φ is 4-I-AVDTC of $\mathcal{M}(P_3)$ and at the same time, $|T_i| = 4$, $i = 1, 2, \cdots, 4$. Thus, φ is a 4-I-AVDETC of $\mathcal{M}(P_3)$.

Case 3 If $n = 4$, as a result of the structure of $\mathcal{M}(P_n)$, we get that $\chi^i_{aet}(\mathcal{M}(P_n)) \geq 5$ according to lemma 1.1. To prove that $\chi^i_{aet}(\mathcal{M}(P_n)) = 5$, only we need to prove that $\mathcal{M}(P_n)$ has a 5-I-AVDETC.

We get a total coloring φ for $\mathcal{M}(P_n)$ as follows.

$\varphi(t_i) = i, i = 1, 2, 3, \varphi(t_4) = 5, \varphi(w) = 5;$

$\varphi(s_i) = i, i = 1, 2, 3, 4, \varphi(t_i t_{i+1}) = i+1, i = 1, 2, 3;$

$\varphi(t_i s_{i+1}) = i+2, i = 1, 2, 3, \varphi(t_i s_{i-1}) = i-1, i = 2, 3, 4;$

$\varphi(s_i w) = 6 - i, i = 1, 2, \varphi(s_i w) = i - 2, i = 3, 4.$

The φ is 4-I-AVDTC of $\mathcal{M}(P_4)$, moreover,

$|T_i| = 4$, $i = 1, 4, 5$, $|T_i| = 5$, $i = 2, 3$. Certainly, φ is a 5-I-AVDETC of $\mathcal{M}(P_4)$.

Case 4 If $n \geq 5$, as a result of the structure of $\mathcal{M}(P_n)$, we get that $\chi^i_{aet}(\mathcal{M}(P_n)) \geq n$ according to lemma 1.1. To prove that $\chi_{aet}^{i}(\mathcal{M}(P_n)) = n$, only we need to prove that $\mathcal{M}(P_n)$ has a n-I-AVDETC.

We get a total coloring φ for $\mathcal{M}(P_n)$ as follows.

$\varphi(t_i) = i, i = 1, 2,\cdots,n, \varphi(w) = n, \varphi(s_i) = i, i = 1, 2,\cdots, n-1, \varphi(s_n) = n;$

$\varphi(t_i t_{i+1}) = i+1, i = 1, 2,\cdots, n-1; \varphi(t_i s_{i+1}) = i+2, i = 1, 2,\cdots, n-2,$

$\varphi(t_{n-1} s_n) = 1, \varphi(t_i s_{i-1}) = i-1, i = 2, 3,\cdots, n,$

$\varphi(s_i w) = n-i+1, i = 1, 2, \varphi(s_i w) = i-2, i = 3, 4,\cdots, n.$

We now show the φ is n-I-AVDTC of $\mathcal{M}(P_n)$

$C(t_1) = \{1, 2, 3\}, \ \overline{C}(w) = \emptyset, C(t_i) = \{i-1, \ i, i+1, i+2\}, i = 2, 3, \cdots, n-2,$

$C(t_{n-1}) = \{1, n-2, n-1, n\}, C(t_n) = \{n-1, n\}, C(s_1) = \{1, n\},$

$C(s_2) = \{2, 3, 4\}, C(s_n) = \{1, n-2\}, C(s_i) = \{i-2, \ i, i+1\}, i = 3, 4, \cdots, n-1;$

meanwhile, $|T_i| = 5, i = 2, n$, and $|T_i| = 6, i = 1, 3, \cdots, n-1$.

So, the φ is a n-I-AVDETC of $\mathcal{M}(P_n)$.

Theorem 2.2 *Suppose the Mycielski graph of a cycle* C_n *is* $\mathcal{M}(C_n)$, *then*

$$\chi_{aet}^{i}(\mathcal{M}(C_n)) = \begin{cases} 5, \ n = 3,4; \\ n, \ n \geq 5. \end{cases} \quad (4)$$

Proof. Suppose the vertex set and edge set of $\mathcal{M}(C_n)$ are that $V(\mathcal{M}(C_n)) = \{t_1, \ t_2, \cdots, t_n, s_1, s_2, \cdots, s_n, w\}$ and $E(\mathcal{M}(C_n)) = E(\mathcal{M}(P_n)) \cup \{t_1 t_n, t_1 s_n, t_n s_1\}$, respectively. We discuss in the following four cases.

Case 1 If $n = 3$, in virtue of the structure of $\mathcal{M}(C_n)$, we get that $\chi_{aet}^{i}(\mathcal{M}(C_n)) \geq 5$ according to lemma 1.1. To prove that $\chi_{aet}^{i}(\mathcal{M}(C_n)) = 5$, only we need to prove that $\mathcal{M}(C_n)$ has a 5-I-AVDETC.

Now we construct a mapping φ from $T(\mathcal{M}(C_3))$ to $\{1, \ 2, \ 3, \ 4, \ 5\}$. Since $E(\mathcal{M}(C_n)) = E(\mathcal{M}(P_n)) \cup \{t_1 t_n, t_1 s_n, t_n s_1\}$, we can acquire a φ based on the φ of $\mathcal{M}(P_3)$ in Theorem 2.1. Firstly, we let

$\varphi(t_1 t_3) = 5, \varphi(t_1 s_3) = 4, \varphi(t_3 s_1) = 1;$

Secondly, we adjust the colors of w, $s_2 w$ and $s_3 w$ to be

$\varphi(s_2 w) = 1, \varphi(s_3 w) = 5, \varphi(w) = 5;$

The colors of other elements are the same as the result of the φ for $\mathcal{M}(P_3)$ in Theorem 2.1, then the adjusted φ is 5-I-AVDTC of $\mathcal{M}(C_3)$ and at the same time, $|T_i| = 4, i = 1, 2, \cdots, 4, |T_5| = 3$. Thus, φ is a 5-I-AVDETC of $\mathcal{M}(C_3)$.

Case 2 If $n = 4$, due to the structure of $\mathcal{M}(C_n)$, we get that $\chi_{aet}^{i}(\mathcal{M}(C_n)) \geq 5$ according to lemma 1.1. To prove that $\chi_{aet}^{i}(\mathcal{M}(C_n)) = 5$, only we need to prove that there exists a 5-I-AVDETC for $\mathcal{M}(C_4)$.

We get a total coloring φ for $\mathcal{M}(C_4)$ by adjusting the φ of $\mathcal{M}(P_4)$, let

$\varphi(t_1 t_4) = 1, \varphi(t_1 s_4) = 4, \varphi(t_4 s_1) = 5;$

meanwhile, swap the colors of $t_2 t_3$ and $t_3 s_4$, thus the new φ is 5-I-AVDTC of $\mathcal{M}(C_4)$, moreover, $|T_i| = 5$ for all $i = 1, 2, 3, 4, 5$. Certainly, φ is a 5-I-AVDETC of $\mathcal{M}(C_4)$.

Case 3 If $n \geq 5$, as a result of the structure of $\mathcal{M}(C_n)$, we get that $\chi_{aet}^{i}(\mathcal{M}(C_n)) \geq n$ according to lemma 1.1. To prove that $\chi_{aet}^{i}(\mathcal{M}(C_n)) = n$, only we need to prove that $\mathcal{M}(C_n)$ has a n-I-AVDETC.

We can get a total coloring φ for $\mathcal{M}(C_n)$ by the φ of $\mathcal{M}(P_n)$, directly. We only let the additional edges be coloring that

$\varphi(t_1t_n) = 1, \varphi(t_1s_n) = n, \varphi(t_ns_1) = 2;$

We show easily that the φ is n-I-AVDTC of $\mathcal{M}(C_n)$, for that compared with the results of $\mathcal{M}(P_n)$ only four vertices' sets are changed

$C(t_1) = \{1, 2, 3, n\}, C(t_n) = \{1, 2, n-1, n\},$

$C(s_1) = \{1, 2, n\}, C(s_n) = \{1, n-2, n\};$

at the same time, $|T_1| = 7$, $|T_i| = 6$, $i = 2, 3, \cdots, n$.

So, the φ is a n-I-AVDETC of $\mathcal{M}(C_n)$.

Theorem 2.3 *Suppose the Mycielski graph of a fan* F_n *is* $\mathcal{M}(F_n)$, *then*

$$\chi_{aet}^{i}(\mathcal{M}(F_n)) = \begin{cases} 7, & n = 3; \\ 2n, & n \geq 4. \end{cases} \tag{5}$$

Proof. Suppose the vertex set and edge set of $\mathcal{M}(F_n)$ are that $V(\mathcal{M}(F_n)) = \{t_0, t_1, t_2, \cdots, t_n, s_1, s_0, s_2, \cdots, s_n, w\}$ and $E(\mathcal{M}(F_n)) = \{t_0t_j, t_it_{i+1}, s_jw, s_0w \mid i = 1, 2, \cdots, n-1, j = 1, 2, \cdots, n\} \cup \{t_is_j \mid t_it_j \in E(F_n), i = 0, 1, 2, \cdots, n, j = 1, 2, \cdots, n\}$, respectively. We discuss in the following two cases.

Case 1 If $n = 3$, on account of the structure of $\mathcal{M}(F_n)$, we get that $\chi_{aet}^{i}(\mathcal{M}(F_n)) \geq 7$ according to lemma 1.1. To prove that $\chi_{aet}^{i}(\mathcal{M}(F_n)) = 7$, only we need to prove that $\mathcal{M}(F_n)$ has a 7-I-AVDETC.

Now we construct a mapping from $T(\mathcal{M}(F_3))$ to $\{1, 2, \cdots, 7\}$ as follows

$\varphi(t_i) = i, i = 1, 2, 3, \varphi(t_0) = 7, \varphi(w) = 7;$

$\varphi(s_0) = 4, \varphi(s_1) = 1, \varphi(s_2) = 5, \varphi(s_3) = 6;$

$\varphi(t_1t_2) = 2, \varphi(t_2t_3) = 3, \varphi(t_0t_i) = i + 3$, i = 1, 2, 3;

$\varphi(t_1s_2) = 6, \varphi(t_2s_3) = 7, \varphi(t_3s_2) = 5, \varphi(t_2s_1) = 4;$

$\varphi(s_0t_i) = i + 4, i = 1, 2, 3; \varphi(s_0w) = i + 1, i = 0, 1, 2, 3.$

The φ is a 4-I-AVDTC of $\mathcal{M}(F_3)$ and meanwhile, $|T_i| = 4$, $i = 1, 2, \cdots, 7$. Thus, φ is a 7-I-AVDETC of $\mathcal{M}(F_3)$.

Case 2 If $n \geq 4$, since $\mathcal{M}(F_n)$ has only one maximum degree vertex t_0, we get that $\chi_{aet}^{i}(\mathcal{M}(F_n)) \geq 2n$ according to definition 1.1. To prove that $\chi_{aet}^{i}(\mathcal{M}(F_n)) = 2n$, only we need to prove that $\mathcal{M}(P_n)$ has a $2n$-I-AVDETC.

We obtain a total coloring φ for $\mathcal{M}(F_n)$ as follows.

$\varphi(t_0) = \varphi(w) = 2n, \varphi(t_i) = i, i = 1, 2, \cdots, n-1, \varphi(t_n) = n-1;$

$\varphi(s_0) = n, \varphi(s_1) = 1, \varphi(s_i) = i + n - 1, i = 2, 3, \cdots, n;$

$\varphi(t_it_{i+1}) = i + 1, i = 1, 2, \cdots, n-1; \varphi(t_0t_i) = i + n,$

$\varphi(t_0s_i) = i, \quad i = 1, 2, \cdots, n; \varphi(s_0t_i) = i + n - 1, \quad i = 1, 2, \cdots, n;$

$\varphi(t_is_{i-1}) = i + n - 2, i = 2, 3, \cdots, n, \varphi(t_is_{i+1}) = i + n + 1, i = 1, 2, \cdots, n-1,$

$\varphi(s_iw) = n - i + 1, i = 1, 2, \cdots, n-2, n, \varphi(s_{n-1}w) = 2n.$

We now show the φ is n-I-AVDTC of $\mathcal{M}(F_n)$

$\overline{C}(t_0) = \emptyset$, $C(t_1) = \{1, 2, n, n+1, n+2\}$,$\overline{C}(w) = \{n,\ n+2,\ n+3,\cdots, 2n-1\}$,

$C(t_i) = \{\ i,\ i+1,\ i+n-2, i+n-1,\ i+n\}$, $i = 2, 3, \cdots, n-1$,
$C(t_n) = \{n,\ 2n-2,\ 2n-1,\ 2n\}$; $C(s_0) = \{1,\ n,\ n+1,\ \cdots, 2n-1\}$,
$C(s_1) = \{1, 2,\ n\}$, $C(s_i) = \{i, i+1, i+n-1,\ i+n\}$, $i = 2, 3, \cdots, n-2,\ n$,
$C(s_{n-1}) = \{n-1, 2n-2, 2n-2,\ 2n\}$;
moreover, $|T_i| = 4$, $i = 1, 2, \cdots, n-1$, $|T_i| = 5$, $i = n, n+1,\ \cdots, 2n$.

So, the φ is a $2n$-I-AVDETC of $\mathcal{M}(F_n)$.

Theorem 2.4 *Suppose the Mycielski graph of a wheel* F_n *is* $\mathcal{M}(W_n)$, *then*

$$\chi_{aet}^{i}(\mathcal{M}(W_n)) = \begin{cases} 7, & n = 3; \\ 2n, & n \geq 4. \end{cases} \quad (6)$$

Proof. Suppose the vertex set and edge set of $\mathcal{M}(W_n)$ are that $V(\mathcal{M}(W_n))= \{t_0,\ t_1,\ t_2, \cdots,\ t_n,\ s_0,\ s_1,\ s_2, \cdots, s_n, w\}$ and $E(\mathcal{M}(W_n)) = E(\mathcal{M}(F_n)) \cup \{t_1t_n,\ t_1s_n,\ t_ns_1\}$, respectively. First of all, we deduce that $\chi_{aet}^{i}(\mathcal{M}(W_3)) \geq 7$ and $\chi_{aet}^{i}(\mathcal{M}(W_n)) \geq 2n$ for $n \geq 4$ according the structure of $\mathcal{M}(W_n)$ using the ahead theorem method. To prove the equation (6) to be true, we can easily construct a mapping φ such that φ is a 7-I-AVDETC of $\mathcal{M}(W_3)$ or a $2n$-I-AVDETC of $\mathcal{M}(W_n)$. Since $E(\mathcal{M}(W_n)) = E(\mathcal{M}(F_n)) \cup \{t_1t_n,\ t_1s_n,\ t_ns_1\}$, we color the additional three edges $t_1t_n,\ t_1s_n,\ t_ns_1$ with colours 1, 3, 4, respectively, and keep the same coloring of other elements with the φ of $\mathcal{M}(F_n)$ in theorem 2.3, then we can test and verify the updating φ satisfying that which is a 7-I-AVDETC of $\mathcal{M}(W_3)$ or a $2n$-I-AVDETC of $\mathcal{M}(W_n)$, and $|T_i| = 4$, $i = 2, 5, 6\cdots, n-1$, $|T_i| = 5$, $i = 1, 3, 4, n, n+1,\ \cdots, 2n$. Here we might note that when i are same number, the $|T_i|$ is assigned to the later. Thus, the conclusion (6) is obviously true.

3. Conclusion

In this paper, we investigate the incidence-adjacent vertex distinguishing equitable total coloring of Mycielski graphs of path, cycle, fan, wheel and of which the incidence-adjacent vertex distinguishing equitable total chromatic numbers are confirmed. We push ahead with the above work by the following conclusion for the incidence-adjacent vertex distinguishing equitable total chromatic numbers of Mycielski graphs. If the a Mycielski graph($\mathcal{M}(G)$) has only one maximum degree vertex or two and even more maximum degree vertices which are not adjacent, then $\chi_{aet}^{i}(\mathcal{M}(G)) = \Delta + 1$; if it has at least two adjacent maximum degree vertices, then $\chi_{aet}^{i}(\mathcal{M}(G)) = \Delta + 2$.

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